Dirac作用素の指数の局所化

11月8日の幾何

藤田 玄

(日本女子大学理学部)

自己紹介

藤田 玄 (ふじた はじめ)

- 2002年 東大数理 修士課程 入学
- 2007年 東大数理にて学位取得 (主査：古田幹雄)
- 学位論文 :「幾何的量子化・共形場理論・(2+1)次元位相的
  場の理論における対称性の研究 」
- 研究分野：シンプレクティック幾何学(幾何学的量子化/Hamiltonトーラス作用)
  情報幾何学、トーリックケーラー多様体のGH収束

古田先生は
私の大学院時代の指導教員です。

古田先生は
ゲージ理論と低次元トポロジー
への応用を研究する数学者です。

古田先生は

私の共同研究者です。吉田尚彦さんと3人でDirac作用素の指数の局所化に関する論文を書きました。

共同研究は2008年2月25日の古田先生からのメールで始まりました。

real polarizationとcomplex polarizationの関係についてですが、
吉田君のやっている議論に触発されて、
「BS leafとsingular leafの（重み付きの）個数が
Riemann-Roch数に等しい」
ということを、局所化を使って導く道筋を考えてみました。
細部は詰めていません。

藤田
Riemann面上のベクトル束のモジュライの幾何学的量子化の偏極依存性

real polarizationとcomplex polarizationの関係についてですが、
吉田君のやっている議論に触発されて、
「BS leafとsingular leafの（重み付きの）個数が
Riemann-Roch数に等しい」
ということを、局所化を使って導く道筋を考えてみました。
細部は詰めていません。

吉田さん
トーリック多様体(の一般化)に対する“RR=BS”

この研究の核となるアイデアはいわゆる

Dirac作用素のWitten deformation

のある変種を用いることにあります。

このWitten deformationというテクニックは古田先生の著書「指数定理」においても重要な位置を占めています。

古田先生は2025年1月にあるところで次のような発言を残されています。

==========

私にとってたとえば Witten deformation は「趣味」でした。それが自分の仕事とはじめて結びついたのは [古田・藤田・吉田] においてでした。

「趣味」がどのような形に芽吹くのか、あるいは他の営みのよき導きとなるのか、あるいはわくわくする子供の目をいつもよみがえらせるものとしてあるのか、それは様々だと思います。

==========

実は私はこのテーマで古田先生と同じ研究集会に参加したことがありません。

==========

私にとってたとえば Witten deformation は「趣味」でした。それが自分の仕事とはじめて結びついたのは [古田・藤田・吉田] においてでした。

「趣味」がどのような形に芽吹くのか、あるいは他の営みのよき導きとなるのか、あるいはわくわくする子供の目をいつもよみがえらせるものとしてあるのか、それは様々だと思います。

==========

古田先生はこのテーマに関して外部で講演したこともほぼないはずです。

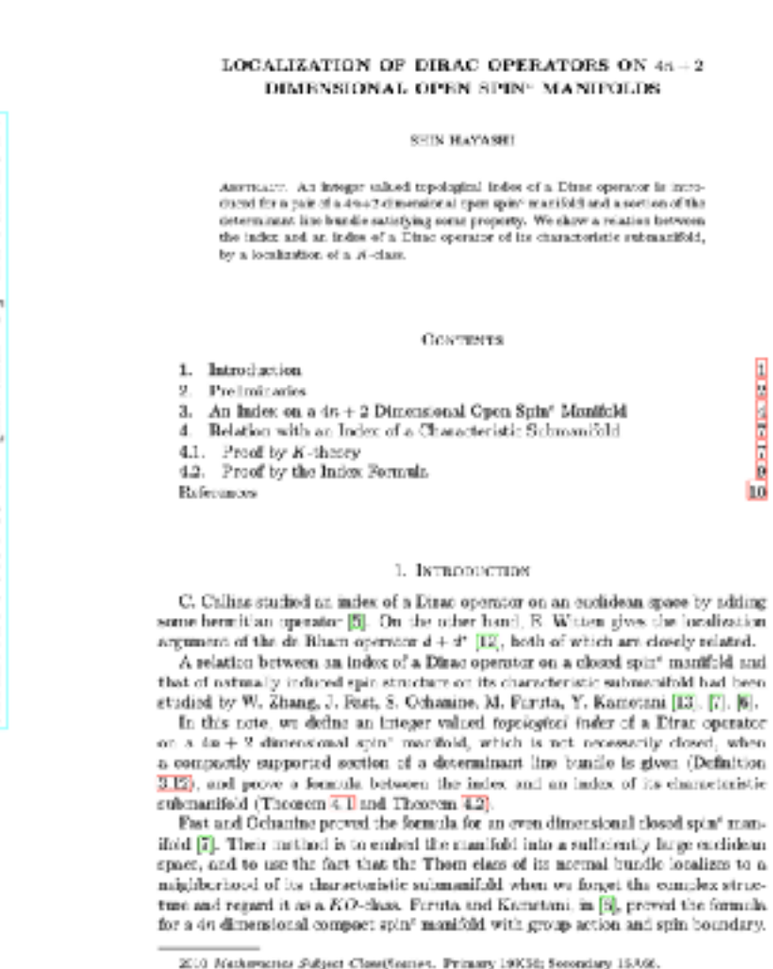
(2010年度日本数学会年会トポロジー分科会総合講演「局所化とトーラス束」のみ?)

依頼があっても私や吉田さんを講演者として代わりに推薦して下さいました。

古田先生にはこの「趣味」の「本業」(さらには格子ゲージ理論)へのフィードバックの展望もあったと思われます。

このような形で古田先生の趣味に関われたことは
私の人生でかけがえのない経験になりました。

その後も古田先生の周辺の若い方達により局所化が
背景にあるような仕事が断続的に産み出されました。



LOCALIZATION OF DIRAC OPERATORS ON $4n-2$ DIMENSIONAL OPEN SPIN^c MANIFOLDS

SHIN HAYASHI

Abstract. An integer valued topological index of a Dirac operator is introduced for a pair of a $4n+2$ dimensional open spin^c manifold and a section of the determinant line bundle satisfying some property. We show a relation between the index and an index of a Dirac operator of its characteristic submanifold, by a localization of a K-class.

Contents

1. Introduction 1
2. Preliminaries 2
3. An Index on a $4n+2$ Dimensional Open Spin^c Manifold 3
4. Relation with an Index of a Characteristic Submanifold 7
4.1. Proof by K-theory 7
4.2. Proof by the Index Formula 9
References 10

1. Introduction

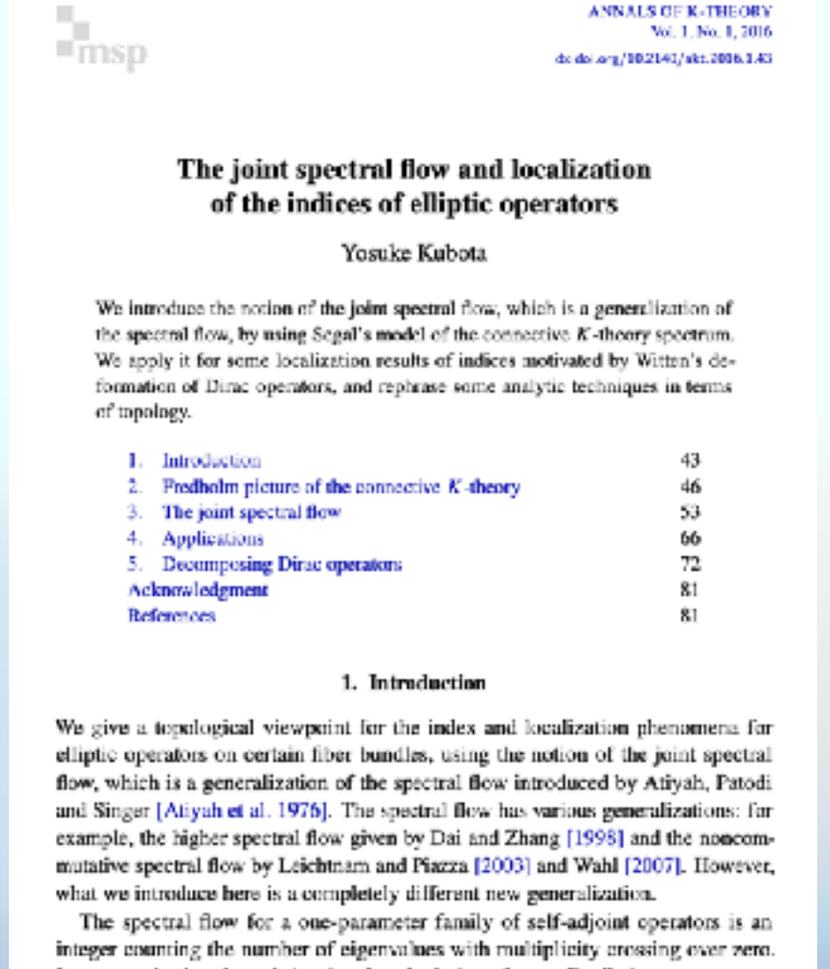
C. Callias studied an index of a Dirac operator on an euclidean space by adding some hermitian operator [5]. On the other hand, E. Witten gives the localization argument of the de Rham operator $d+d^*$ [12], both of which are closely related.

A relation between an index of a Dirac operator on a closed spin^c manifold and that of naturally induced spin structure on its characteristic submanifold had been studied by W. Zhang, J. Fust, S. Ochanine, M. Furuta, Y. Kametani [13], [7], [6].

In this note, we define an integer valued topological index of a Dirac operator on a $4n+2$ dimensional spin^c manifold, which is not necessarily closed, when a compactly supported section of a determinant line bundle is given (Definition 3.12), and prove a formula between the index and an index of its characteristic submanifold (Theorem 4.1 and Theorem 4.2).

Fust and Ochanine proved the formula for an even dimensional closed spin^c manifold [7]. Their method is to embed the manifold into a sufficiently large euclidean space, and to use the fact that the Thom class of its normal bundle localizes to a neighborhood of its characteristic submanifold when we forget the complex structure and regard it as a KO-class. Furuta and Kametani, in [6], proved the formula for a $4n$ dimensional compact spin^c manifold with group action and spin boundary.

2010 *Mathematics Subject Classification.* Primary 19K56; Secondary 15A66.



msp

ANNALS OF K-THEORY
Vol. 1, No. 1, 2016
dx.doi.org/10.2140/akt.2016.1.43

The joint spectral flow and localization of the indices of elliptic operators

Yosuke Kubota

We introduce the notion of the joint spectral flow, which is a generalization of the spectral flow, by using Segal's model of the connective K-theory spectrum. We apply it for some localization results of indices motivated by Witten's deformation of Dirac operators, and rephrase some analytic techniques in terms of topology.

1. Introduction 43
2. Predholm picture of the connective K-theory 46
3. The joint spectral flow 53
4. Applications 66
5. Decomposing Dirac operators 72
Acknowledgment 81
References 81

1. Introduction

We give a topological viewpoint for the index and localization phenomena for elliptic operators on certain fiber bundles, using the notion of the joint spectral flow, which is a generalization of the spectral flow introduced by Atiyah, Patodi and Singer [Atiyah et al. 1976]. The spectral flow has various generalizations: for example, the higher spectral flow given by Dai and Zhang [1998] and the noncommutative spectral flow by Leichtnam and Piazza [2003] and Wahl [2007]. However, what we introduce here is a completely different new generalization.

The spectral flow for a one-parameter family of self-adjoint operators is an integer counting the number of eigenvalues with multiplicity crossing over zero.



Communications in Mathematical Physics

Digital Object Identifier (DOI): https://doi.org/10.1007/s00220-019-03593-1

A Topological Approach to Indices of Geometric Operators on Manifolds with Fibered Boundaries

Mayuko Yamashita

Research Institute for Mathematical Sciences, Kyoto University, Oiwake-cho, Kitashirakawa, Sakyo-ku, Kyoto 606-8502, Japan. E-mail: mayuko@kurims.kyoto-u.ac.jp

Received: 11 February 2019 / Accepted: 25 August 2019
Published online: 3 November 2019 – © The Author(s) 2019

Abstract. In this paper, we investigate topological aspects of indices of twisted geometric operators on manifolds equipped with fibered boundaries. We define K-groups relative to the pull-back and the boundary fibration, and show that indices of twisted geometric operators, defined by complete Φ-or edge metrics, can be regarded as the index pairing over these K-groups. We also prove various properties of these indices using groupoid deformation techniques. Using these properties, we give an application to the localization problem of signature operators for singular fiber bundles.

Contents

1. Introduction . . . 58
2. Preliminaries . . . 62
2.1 Representable K-theory . . . 62
2.2 $\mathcal{O}_1$-invertible perturbations . . . 63
2.3 Groupoids . . . 66
2.4 b, Φ- calculus and corresponding groupoids . . . 69
3. Indices of Geometric Operators on Manifolds with Fibered Boundaries: The Case Without Perturbations . . . 98
3.1 The definition of indices . . . 98
3.2 Properties . . . 101
3.3 The cases of twisted spin^c and signature operators . . . 106
4. Indices of Geometric Operators on Manifolds with Fibered Boundaries: The Case with Fiberwise Invertible Perturbations . . . 109
4.1 The general situation . . . 110

このような形で古田先生の趣味に関われたことは
私の人生でかけがえのない経験になりました。

その後も古田先生の周辺の若い方達により局所化が
背景にあるような仕事が断続的に産み出されました。

arXiv:1912.07994v3 [math.DG] 2 May 2023

SPECTRAL CONVERGENCE IN GEOMETRIC QUANTIZATION — THE CASE OF NON-SINGULAR LANGRANGIAN FIBRATIONS

KOTA HATTORI AND MAYUKO YAMASHITA

Abstract. This paper is a sequel to [21]. We develop a new approach to geometric quantization using the theory of convergence of metric measure spaces. Given a family of Kähler polarizations converging to a non-singular real polarization on a prequantized symplectic manifold, we obtain the spectral convergence result of $\overline{\partial}$-Laplacians, as well as the convergence result of quantum Hilbert spaces. We also consider the case of almost Kähler quantization for compatible almost complex structures, and show the analogous convergence results.

Contents

1. Introduction 2
2. Settings 5
2.1. Almost complex structures 5
2.2. Bohr-Sommerfeld fibers 8
2.3. Polarizations 8
2.4. Local descriptions 10
2.5. The structure of the limit spaces 12
3. Spectral convergence 14
3.1. Convergence of spectral structures 14
3.2. Lie group actions on Spectral structures 16
3.3. Strong spectral convergence of equivariant Laplacians 17
3.4. Ricci curvature 19
4. The compact spectral convergence 21
4.1. A local estimate 21
4.2. Localization of $H^{1,2}$-bounded functions to Bohr-Sommerfeld fibers 23
4.3. Convergence of $H^{1,2}$-bounded sequences 27
5. Convergence of quantum Hilbert spaces 28
5.1. Almost Kähler quantization 28
5.2. Convergence of the quantum Hilbert spaces 33
6. Examples 33
6.1. Kähler quantization 33
6.2. Almost Kähler quantization 33

msp

Algebraic & Geometric Topology 25:2 (2025) 887–918
DOI: 10.2140/agt.2025.25.887
Published: 16 May 2025

Localization of a $KO^*(pt)$-valued index and the orientability of the $Pin^-(2)$ monopole moduli space

Jin Miyazawa

It is known that the Dirac index of a Spin^c structure is localized to the characteristic submanifold. We introduce the notion of $G^{\pm}(n,s^+,s^-)$ structure on a manifold as a common generalization of the Spin^c structure and the $H_s(n)$ structure defined by D Freed and M Hopkins, and formulate a version of characteristic submanifold for the $G^-(n,s^+,s^-)$ structure. We show that the $KO^*(pt)$-valued index associated with the $G^{\perp}(n,s^+,s^-)$ structure is localized to the characteristic submanifold. As an application, we give a topological sufficient condition for the moduli space of $Pin^-(2)$ monopoles to be orientable.

57K41, 58J20

1. Introduction 887
2. Definition of the index 889
3. Proof of Main Theorem 894
4. Examples 904
5. The orientability of $Pin^-(2)$-monopole moduli space 908
Appendix 915
References 917

1 Introduction

In this paper, we introduce the notion of $G^{\pm}(n,s^+,s^-)$ structure, which is a generalization of the Spin structure, the Spin^c structure, the $Pin^{\pm}$ structure, and the $H_s(n)$ structure due to Freed and Hopkins [5]. We construct an elliptic differential operator associated with the $G^{\pm}(n,s^-,s^-)$ structure s, and we have its index $\mathrm{ind}(s)$ with values in $KO^{n-s^+-s^-}(pt)$. The index $\mathrm{ind}(s)$ is a generalization of the Atiyah–Milnor–Singer invariant which is defined by the Dirac type operator with the Clifford action for a Spin manifold.

このWitten deformation型の局所化のテクニックは松尾さん山下さんと深谷さんら物理の方々との共同研究でも有効に使われました。

arXiv:2602.12576v2 [math.DG] 10 Apr 2026

CAPTURING THE ATIYAH–PATODI–SINGER INDEX FROM THE LATTICE

SHOTO AOKI, HAJIME FUJITA, HIDENORI FUKAYA, MIKIO FURUTA, SHINICHIROH MATSUO, TETSUYA ONOGI, AND SATOSHI YAMAGUCHI

Abstract: We construct a formulation of the Atiyah-Patodi-Singer index of Dirac operators in lattice gauge theory for domains with compact boundaries in a flat torus. The key idea is to exploit its equality to the spectral flow of the domain wall fermion Dirac operators, which we generalize in this work to cases without product structure near the boundary. We prove that, for sufficiently small lattice spacings, this formulation correctly captures the continuum Atiyah–Patodi–Singer index.

Contents

1. Introduction 2
2. Wilson Dirac operators on a lattice 5
2.1. Dirac operators in continuum space 5
2.2. Wilson Dirac operators 6
3. From lattice to continuum 8
3.1. Finite element interpolator 8
3.2. Properties 9
4. domain-wall fermion Dirac operators and the Atiyah-Patodi-Singer index 10
4.1. Dirac operators on manifolds with boundaries and the APS index 10
4.2. APS index without product structure near the boundary 11
4.3. Domain-wall fermion Dirac operators 12
4.4. The domain-wall Dirac operators and the APS index 14
4.5. domain-wall fermion Dirac operators on a flat torus 16
5. K-groups and unbounded selfadjoint operators 16
5.1. Definitions of $K^{p,q}_\bullet$-cocycles 17
5.2. Set-theoretic techniques 20
5.3. Definition of the semigroup $K^{p,q}_\bullet$ 22
5.4. Group structure on the semigroup $K^{p,q}_\bullet$ 23
5.5. Degrees 25
5.6. Riesz continuous families 26
5.7. The inverse element 28

Preprint number: OU-HEP-1330.

Commun. Math. Phys. 380, 1295–1311 (2020)
Digital Object Identifier (DOI) https://doi.org/10.1007/s00220-020-03806-0

Communications in Mathematical Physics

Check for updates

The Atiyah–Patodi–Singer Index and Domain-Wall Fermion Dirac Operators

Hidenori Fukaya[1], Mikio Furuta[2], Shinichiroh Matsuo[3], Tetsuya Onogi[1], Satoshi Yamaguchi[1], Mayuko Yamashita[4]

[1] Department of Physics, Osaka University, Osaka, Japan.
E-mail: hfukaya@het.phys.sci.osaka-u.ac.jp; onogi@het.phys.sci.osaka-u.ac.jp; yamaguch@het.phys.sci.osaka-u.ac.jp
[2] Graduate School of Mathematical Sciences, The University of Tokyo, Tokyo, Japan.
E-mail: furuta@ms.u-tokyo.ac.jp
[3] Graduate School of Mathematics, Nagoya University, Nagoya, Japan.
E-mail: shinichiroh@math.nagoya-u.ac.jp
URL: https://www.math.nagoya-u.ac.jp/shinichiroh/
[4] Research Institute for Mathematical Sciences, Kyoto University, Kyoto, Japan.
E-mail: mayuko@kurims.kyoto-u.ac.jp

Received: 13 October 2019 / Accepted: 27 April 2020
Published online: 12 August 2020 – © Springer-Verlag GmbH Germany, part of Springer Nature 2020

Abstract: We introduce a *mathematician-friendly* formulation of the *physicist-friendly* derivation of the Atiyah–Patodi–Singer index. In a previous paper, motivated by the study of lattice gauge theory, the physicist half of the authors derived a formula expressing the Atiyah–Patodi–Singer index in terms of the eta invariant of *domain-wall fermion Dirac operators* when the base manifold is a flat 4-dimensional torus. In this paper, we generalise this formula to any even dimensional closed Riemannian manifolds, and prove it mathematically rigorously. Our proof uses a Witten localisation argument combined with a devised embedding into a cylinder of one dimension higher. Our viewpoint sheds some new light on the interplay among the Atiyah–Patodi–Singer boundary condition, domain-wall fermions, and edge modes.

1. Introduction

The Atiyah–Patodi–Singer index theorem [1–3], a generalisation of the Atiyah–Singer index theorem to manifolds with boundary, has been attracting attention in condensed matter physics. For example, Witten used it in [23] to describe the bulk-edge correspon-

局所化の考え方は様々な場面で様々な形で現れます。
あえて広辞苑的な説明をすると：

ある対象の知りたい情報が

その対象の一部のみから抽出される

とでもなるでしょうか。具体的には：

ある多様体の不変量が

部分多様体上の計算で求められる

という使われ方が幾何の文脈でよくあります。

この広辞苑的な説明を逆手にとると(?)次のように
言い換えることができます：

その一部から離れた部分から
その情報への寄与はない

あるいは(非常にナイーヴに)：

不変量が部分多様体上で求められるなら
そこから離れたところからの寄与はない

とも言えるかもしれません。

このある種のトートロジカルな

寄与がないところからの寄与がないことを

示すことで局所化を導く

というストーリーをDirac作用素から決まる不変量に対して実現する手法の一つがWitten deformationです。

ここから、幾何学的量子化に由来するモチベーション、Witten deformationの考え方を説明し、３人の共同研究(FFY)の概要を説明します。

Contents

§1. Geometric Quantization

§2. Naive expectation & Observation

§3. Witten deformation

§4. FFY's formulation

§5. Symplectic case

§6. Comments

§1. Geometric Quantization

§1. Geometric Quantization

$$\boxed{(M,\omega) : \text{symp. mfd.} \rightsquigarrow \mathcal{H} = \mathcal{H}_{\mathcal{P}}(M,\omega) : \text{"nice" Hilbert sp.}}$$

- $\exists$ pre-quantizing line b'dle (L, ∇) (· cpt.ness)
- $\mathcal{P}$: polarization
  $\parallel$
  integrable Lag. subb'dle of $TM \otimes \mathbb{C}$

§1. Geometric Quantization

(M, ω) : symp. mfd. $\rightsquigarrow$ $\mathcal{H} = \mathcal{H}_P(M, \omega)$: "nice" Hilbert sp.

- $\exists$ pre-quantizing line bdle (L, ∇) (+ cpt.ness)
- P : polarization = integrable Lag. subbdle of $TM \otimes \mathbb{C}$

Ex. i) M : Kähler $\Rightarrow P = T^{*}M^{0,1}$: Kähler polarization

$\Rightarrow \mathcal{H}_K := \mathcal{H}_P = H^0(M, L)$: holo. sects of L.

ii) M : Lagrangian $\Rightarrow P = \ker d\pi \otimes \mathbb{C}$: real polarization

$\pi \downarrow$ fibration $\Rightarrow \mathcal{H}_R := \mathcal{H}_P =$ "parallel sects of L on BS-fibers"

B

Question

$\mathcal{H}_P$ depends on $\mathcal{P}$?

e.g. $\mathcal{H}_K \cong \mathcal{H}_R$ in some sense?

§1. Geometric Quantization

Question

$\mathcal{H}_P$ depends on P?

e.g. $\mathcal{H}_K \cong \mathcal{H}_R$ in some sense?

Supporting Facts

$\dim \mathcal{H}_K = \dim \mathcal{H}_R$ for several cases.

- Toric mfd (Danilov)
- Moduli of flat b'dles (Goldman, Jeffrey-Weitsman, ...)
- Flag mfd. (Gelfand-Cetlin, Guillemin-Sternberg, ...)
- Polygon space (Kamiyama, ...)

§1. Geometric Quantization

Kähler polarization & Dirac operator

$$\mathcal{H}_k = H^0(M,L) = \ker(\bar{\partial}_L)$$

$\bar{\partial}_L : [\Omega^0(M,L) \to \Omega^1(M,L)]$

$$H^k(M,L)$$
$$\| \quad \text{Laplacian} : \Omega^k \to \Omega^k$$
$$\ker(\bar{\partial}_L \bar{\partial}_L^* + \bar{\partial}_L^* \bar{\partial}_L)_{(k)}$$
$$\| \quad (\leftarrow \mathbb{Z}/2\text{-grading})$$
$$\ker(\bar{\partial}_L + \bar{\partial}_L^*)^2$$
$$\|$$
$$\ker(\bar{\partial}_L + \bar{\partial}_L^*)$$

Dolbeault-Dirac operator D_L

§1. Geometric Quantization

Kähler polarization & Dirac operator

$$\mathcal{H}_k = H^0(M,L) = \ker(\bar\partial_L)$$

$\bar\partial_L : [\Omega^0(M,L) \to \Omega^1(M,L)]$

$$H^k(M,L) = \ker(\underbrace{\bar\partial_L \bar\partial_L^* + \bar\partial_L^* \bar\partial_L}_{\text{Laplacian} : \Omega^k \to \Omega^k})_{(k)}$$

$$= \ker(\bar\partial_L + \bar\partial_L^*)^2 \quad (\leftarrow \mathbb{Z}/2\text{-grading})$$

$$= \ker(\underbrace{\bar\partial_L + \bar\partial_L^*}_{\text{Dolbeault-Dirac operator } D_L})$$

$$\ker(\bar\partial_L + \bar\partial_L^*)^{\text{even}} - \ker(\bar\partial_L + \bar\partial_L^*)^{\text{odd}} \in K(\text{Pt})$$

$$= \operatorname{ind}(\bar\partial_L + \bar\partial_L^*) \in \mathbb{Z}$$

$$= \operatorname{ind}(D_L)$$ ← depends only on Spin^c-str.

Kähler polarization & Dirac operator

$$\mathcal{H}_k = H^0(M,L) = \ker(\bar\partial_L) \qquad [\Omega^0(M,L) \to \Omega^1(M,L)]$$

$$= \sum_k (-1)^k H^k(M,L)$$

under suitable vanishing assumption of $H^k(M,L)$ $(k \geq 1)$

$$= \ker(\bar\partial_L + \bar\partial_L^*)^{\text{even}} - \ker(\bar\partial_L + \bar\partial_L^*)^{\text{odd}} \in K(\text{Pt})$$

$$= \operatorname{ind}(\bar\partial_L + \bar\partial_L^*) \in \mathbb{Z}$$

$$= \operatorname{ind}(D_L)$$ ← depends only on Spin^c-str.

$H^k(M,L)$

$=$ $\ker(\underbrace{\bar\partial_L \bar\partial_L^* + \bar\partial_L^* \bar\partial_L}_{\text{Laplacian} : \Omega^k \to \Omega^k})_{(k)}$

$=$ ($\leftarrow \mathbb{Z}/2$-grading)

$\ker(\bar\partial_L + \bar\partial_L^*)^2$

$=$

$\ker(\underbrace{\bar\partial_L + \bar\partial_L^*}_{\text{Dolbeault-Dirac operator } D_L})$

§1. Geometric Quantization

Real polarization & BS fiber/orbit

$\pi : M \longrightarrow B$: Lagrangian fib (w/o. sing. fib.)

$\rightsquigarrow \forall b \in B \quad \pi^{-1}(b) \subset M$: Lag. submfd.

$\rightsquigarrow (L, \nabla)|_{\pi^{-1}(b)}$: flat line b'dle

$(curv(\nabla)|_{\pi^{-1}(b)} = \omega|_{\pi^{-1}(b)} = 0)$

$\rightsquigarrow \Gamma^{flat}((L, \nabla)|_{\pi^{-1}(b)}) \cong \begin{cases} \mathbb{C} & (\text{if } (L,\nabla)|_{\pi^{-1}(b)} \text{ has trivial holonomy}) \\ 0 & (\text{o.w.}) \end{cases}$

$(= \text{flat sections. of } (L, \nabla)|_{\pi^{-1}(b)})$

§1. Geometric Quantization

Real polarization & BS point/fiber

Def. $b \in B$ with $\Gamma^{\text{flat}}((L,\nabla)|_{\pi^{-1}(b)}) \cong \mathbb{C}$ is called a Bohr-Sommerfeld point, and $\pi^{-1}(b)$ is called a BS fiber.

Def. $$\mathcal{H}_R := \bigoplus_{b:\text{BS-pt}} \Gamma^{\text{flat}}((L,\nabla)|_{\pi^{-1}(b)})$$

FACT
- $\forall b \in B.\ \pi^{-1}(b) \subset M$: Lag. torus
- B has an integral affine str.
  $\rightsquigarrow B_{\mathbb{Z}}$ makes sense
- BS-pts $= B_{\mathbb{Z}}$

§2. Naive expectation & Observation

§2. Naive expectation & Observation

Suppose: $\exists \pi : M \longrightarrow B$ $\rightsquigarrow$ $\mathcal{H}_R = \bigoplus_{b \in BS} \Gamma^{flat}(L|_{\pi^{-1}(b)})$

$\exists$ Kähler str. $\rightsquigarrow$ $\mathcal{H}_k = H^0(M, L)$

$\mathcal{H}_k = ind(D_L)$

§2. Naive expectation & Observation

Suppose: $\exists\, \pi : M \longrightarrow B$ $\rightsquigarrow$ $\mathcal{H}_R = \bigoplus_{b \in BS} \Gamma^{flat}(L|_{\pi^{-1}(b)})$

$\exists$ Kähler str. $\rightsquigarrow$ $\mathcal{H}_k = H^0(M, L)$

$\mathcal{H}_R \overset{?}{\cong} \mathcal{H}_k$

$\mathcal{H}_k = \mathrm{ind}(D_L)$

If so: $\boxed{\mathrm{ind}(D_L) \cong \bigoplus_{b \in BS} \Gamma^{flat}(L|_{\pi^{-1}(b)})}$ $\rightsquigarrow$ "RR = BS"

(can be formulated in the spin^c-setting)

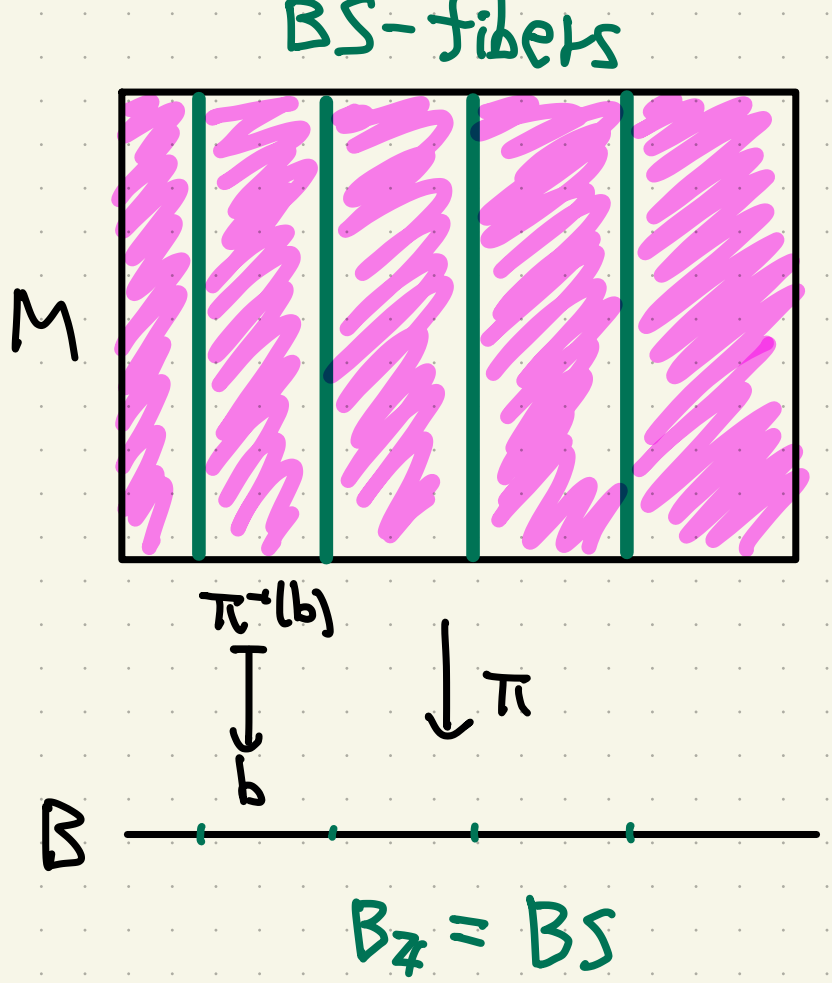
$\mathrm{ind}(D_L)$ called the Riemann-Roch number $RR(M, L)$

$\rightsquigarrow$ Localization of $\mathrm{ind}(D_L)$ to BS-pt.s ??

§2. Naive expectation & Observation

Suppose: $\exists \pi : M \longrightarrow B$ $\rightsquigarrow$ $\mathcal{H}_R = \bigoplus_{b \in BS} \Gamma^{flat}(L|_{\pi^{-1}(b)})$

$\exists$ Kähler str. $\rightsquigarrow$ $\mathcal{H}_K = H^0(M, L) = ind(D_L)$



$ind(D_L) \overset{?}{=} \dim \mathcal{H}_R = \#BS$

Localization of $ind(D_L)$ to BS?

No contribution from ▨ part.

§2. Naive expectation & Observation

Observation $\pi : (M, \omega) \longrightarrow B$: Lagrangian fib.

- $b \notin B_2 = BS\,pt.s \iff \Gamma^{flat}\left((L, \nabla)|_{\pi^{-1}(b)}\right) = 0$

$$\Gamma^{flat}\left((L, \nabla)|_{\pi^{-1}(b)}\right) = H^0\left(\pi^{-1}(b), (L, \nabla)|_{\pi^{-1}(b)}\right)$$

($L|_{\pi^{-1}(b)}$-valued Harmonic fct.s/forms)

$$\Updownarrow$$

$$Ker(D_{\pi,b}) = 0$$

$$\Updownarrow$$

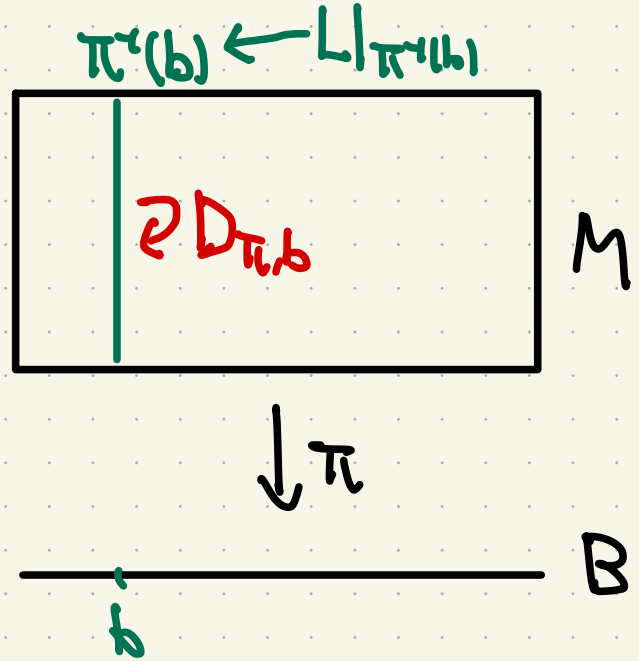
$$Ker(d_{L,b} d_{L,b}^* + d_{L,b}^* d_{L,b})_{(0)}$$

Laplacian

$$\Updownarrow \ (\leftarrow \mathbb{Z}/2\text{-grading})$$

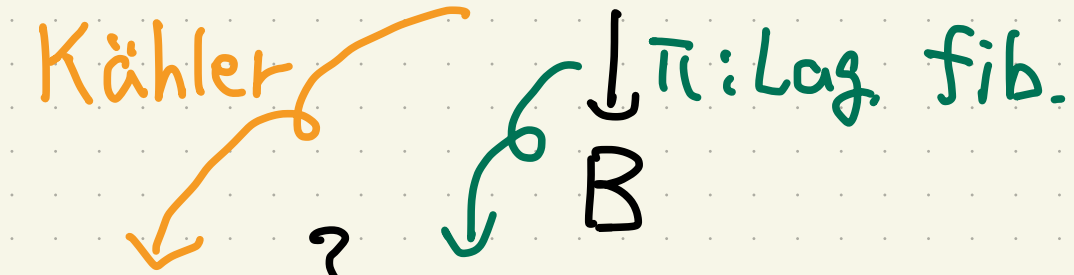
$$Ker(d_{L,b} + d_{L,b}^*)$$

Dirac operator $=: D_{\pi,b}$



Summary

- $(L, \nabla) \longrightarrow (M, \omega)$



- $\mathcal{H}_K \overset{?}{\cong} \mathcal{H}_R \longleftrightarrow$ Localization of $ind(D_L)$ to BS

$\mathcal{H}_K$ = $ind(D_L)$ = RR

$\mathcal{H}_R$ = "counting BS-pts"

- $b \in BS \longleftrightarrow \ker(D_{\pi,b}) \neq 0$

$BS = supp(D_\pi)$

§3. Witten deformation

§3. Witten deformation

Basic setup

- M : Riem. mfd. (may be non-cpt, cplt)
- $W \longrightarrow M$: $\mathbb{Z}/2$-graded Clifford mod. b'dle.

  $W = W^+ \oplus W^-$ i.e., $\exists\, c : T^*M \longrightarrow \mathrm{End}(W)$ (Clifford act.)

  $v \longmapsto c(v)$: odd, skew-adj.

  w/. $c(v)^2 = -|v|^2$.

  [e.g. $W = \Lambda^\bullet T^*M$, $c(\alpha) = \alpha\wedge\bullet - \iota_\alpha(\cdot)$]
- $D : \Gamma(W) \circlearrowright$: Dirac(-type) op.

  i.e., 1-st. order, odd, formally self-adj. op.

  w/. [principal symbol of D] = [Clifford act. c]

  [e.g. $D = d + d^*$

§3. Witten deformation

Suppose $\exists h : W \longrightarrow W$: self-adj., odd.

w/. ⛧ : $c(\alpha)h + h\,c(\alpha) = 0$ $\left(\forall \alpha \in T^*M\right)$.

e.g. $f : M \longrightarrow \mathbb{R}$: Morse fct., $W = \Lambda^\bullet T^*M$

$\rightsquigarrow h = \hat{c}(df) = \sqrt{-1}\left(df\wedge + \iota_{df}(\cdot)\right)$, $h^2 = |df|^2$

← Witten's Morse theory

§3. Witten deformation

Suppose $\exists h : W \longrightarrow W$: self-adj., odd.

w/. $\boxed{\star : c(\alpha)h + h\,c(\alpha) = 0 \quad (\forall \alpha \in T^*M).}$

e.g. $f : M \longrightarrow \mathbb{R}$: Morse fct., $W = \Lambda^\bullet T^*M$

$\rightsquigarrow h = \hat{c}(df) = \sqrt{-1}(df \wedge + \iota_{df}(\cdot))$, $h^2 = |df|^2$

← Witten's Morse theory

Consider the deformation $D_t := D + th$.

Key Claim (vanishing) $\mathrm{Ker}(h_x) = 0 \quad \forall x \in M$

$\Longrightarrow \mathrm{Ker}\, D_t^2 \ (= \mathrm{Ker}\, D_t) = 0 \quad (\forall t \gg 0).$

In particular:

$(\mathrm{ind}(D) =)\ \mathrm{ind}(D_t) = \dim \mathrm{Ker}\, D_t^+ - \dim \mathrm{Ker}\, D_t^- = 0$

§3. Witten deformation

Key Claim $\ker(h_x) = 0 \ \forall x \in M$

$$\Longrightarrow \ker D_t^2 \ (= \ker D_t) = 0.$$

☺ $D_t^2 = D^2 + t(Dh + hD) + t^2 h^2$

$Dh + hD$: order 0 by (★)

$\geq t(Dh + hD) + t^2 h^2$

$\geq -Ct h^2 + t^2 h^2$

h^2: [$\ker h = 0 \Rightarrow \exists C > 0 \ |Dh + hD| \leq Ch^2$]

$= (t^2 - Ct)h^2 > 0 \quad$ for $t \gg 0$. //

§3. Witten deformation

Key Claim

$\Rightarrow$ • $\mathrm{Supp}(h) := \overline{\{x \in M \mid \mathrm{Ker} h_x \neq 0\}}$: compact.

$\Rightarrow \dim \mathrm{Ker} D_t < \infty$.

• $\mathrm{ind}(D_t) := \dim \mathrm{Ker} D_t^+ - \dim \mathrm{Ker} D_t^-$ is well-def.

$(t \gg 0)$

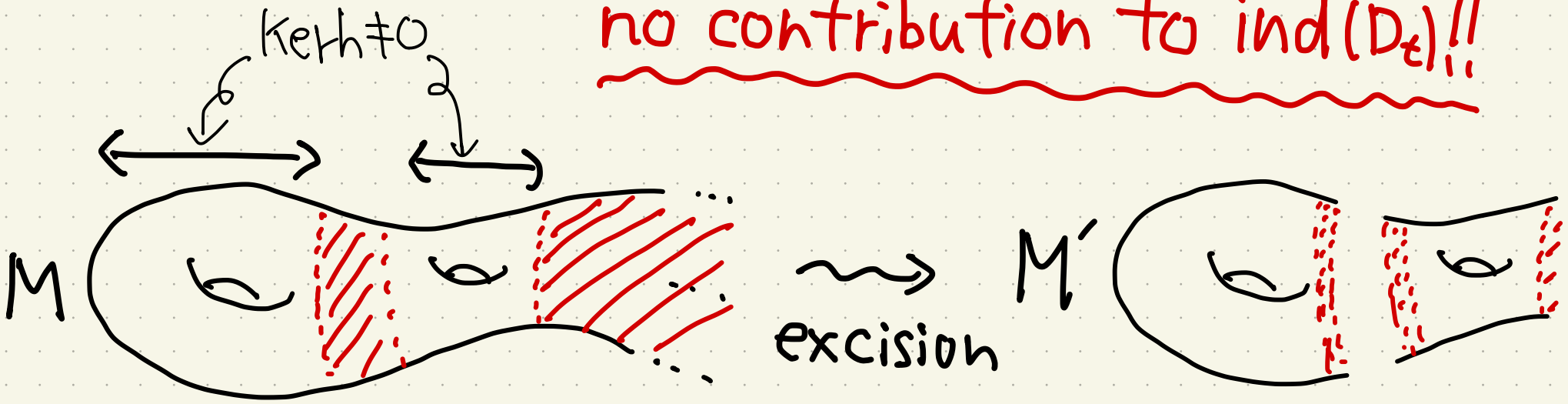
• $\mathrm{ind}(D_t)$ satisfies the excision property.

• M : closed $\Rightarrow \mathrm{ind}(D_t) = \mathrm{ind}(D)$. $(= \mathrm{ind}(d+d^*) = \chi(M))$

↑ topological inv.

§3. Witten deformation

Intuitively : "$\ker h = 0$" part has no contribution to $\mathrm{ind}(D_t)$!!



Excision property

$\updownarrow$

Localization to "$\ker h \neq 0$"

e.g. $h = \hat{c}(df)$ (Morse)

$\rightsquigarrow \mathrm{supp}(h) = df^{-1}(0) = \mathrm{Crit}(f)$

$\Rightarrow \chi(M) \leftrightarrow \mathrm{Crit}(f)$

Summary

- $D_t = D + th$ w/. ✡ $c \cdot h + h \cdot c = 0$
- $\mathrm{Supp}(h)$ $(:= \overline{\{x \in M \mid \mathrm{Ker} h_x \neq 0\}})$ $= \phi \Rightarrow \mathrm{Ker}(D_t) = 0$ $(t \gg 0)$.
- $\mathrm{Supp}(h)$: compact $\Rightarrow \mathrm{ind}(D_t)$: well-def.
- $\mathrm{ind}(D_t)$ localizes at $\mathrm{Supp}(h)$

§4. FFY's formulation

§4. FFY's formulation

Strategy

$D + th$

(Witten deformation)

ind $\int$ localizes at

$\mathrm{Supp}(h)$

$=$

$\{x \in M \mid \ker(h_x) \neq 0\}$

replace

$\Longrightarrow$

h

with

"D_π".

$D + tD_\pi$

ind $\int$ localizes at

$\mathrm{Supp}(D_\pi)$

$=$

"$\ker(D_\pi) \neq 0$"

$=$

BS

§4. FFY's formulation

Basic setup

- M : Riem. mfd. (cp'lt) not necessarily symplectic
- $W \longrightarrow M$: $\mathbb{Z}/2$-gr. Clifford module b'dle
  (i.e., $\exists c : TM \longrightarrow \mathrm{End}(W)$, $c(v)$: skew Herm. odd. $c(v)^2 = -|v|^2$)
- $D : \Gamma(W) = \Gamma(W^+) \oplus \Gamma(W^-) \circlearrowright$: Dirac-type op.
  (i.e., 1-st. order, formaly self-adj., odd,
  [principal symbol of D] = [Clifford act. c])

§4. FFY's formulation

Def. $(V \xrightarrow{\pi} U,\ D_\pi)$: generalized real polarization if:

0) $V \subset M$: open

1) $\pi : V \longrightarrow U$: fiber b'dle w/ closed fiber.
Riem. submersion. $(\rightsquigarrow TV \cong \pi^* TU \oplus T_\pi V)$

2) $D_\pi : \Gamma(W|_V) \circlearrowright$ 1-st order, self-adj, odd.

$$\begin{cases} (2\text{-}1)\ D_\pi \cdot (\pi^*\varphi) = (\pi^*\varphi)\cdot D_\pi \quad \forall \varphi \in C^\infty(U) \\ (2\text{-}2)\ [\text{principal symbol of } D_\pi] = \text{Cliff. act. of } T_\pi V \\ (2\text{-}3)\ D_\pi \cdot c(\tilde{v}) + c(\tilde{v})\cdot D_\pi = 0 \quad \forall v \in \pi^* TU \cong T_\pi V^\perp \end{cases}$$

§4. FFY's formulation

- (2-1) $D_\pi \cdot (\pi^*\varphi) = (\pi^*\varphi) \cdot D_\pi \quad \forall \varphi \in C^\infty(U)$
- (2-2) [principal symbol of D_π] = Cliff act. of $\overline{T_\pi V}$

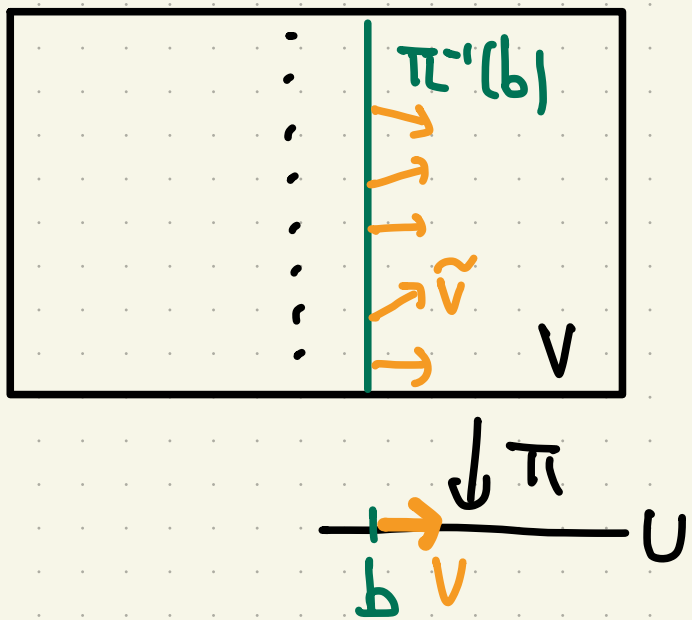
↳ $D_{\pi,b} : \Gamma(W|_{\pi^{-1}(b)}) \circlearrowright$: Dirac op. on $\pi^{-1}(b)$

§4. FFY's formulation

- (2-1) $D_\pi \cdot (\pi^*\varphi) = (\pi^*\varphi) \cdot D_\pi \quad \forall \varphi \in C^\infty(U)$
- (2-2) [principal symbol of D_π] = Cliff act. of $T_\pi V$

↪ $D_{\pi,b} : \Gamma(W|_{\pi^{-1}(b)}) \circlearrowright$: Dirac op. on $\pi^{-1}(b)$

- (2-3) $D_\pi \cdot c(\tilde{v}) + c(\tilde{v}) \cdot D_\pi = 0 \quad \forall v \in \pi^* TU \cong T_\pi V^\perp$



↪ "$\star$: $c \cdot h + h \cdot c = 0$"

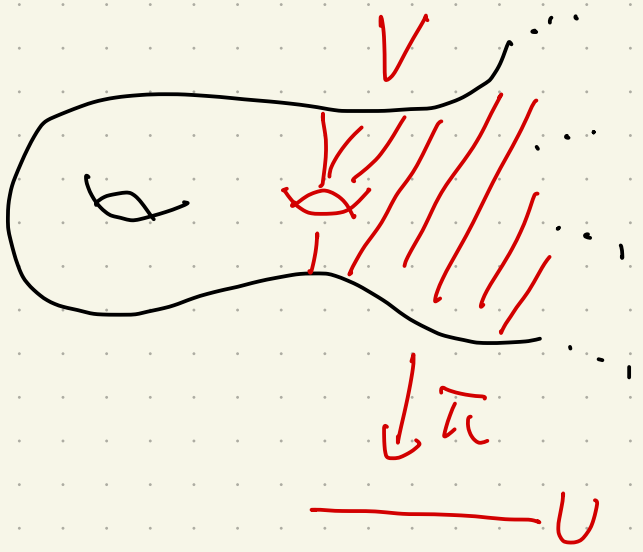
in the Witten deformation

§4. FFY's formulation

Def. $(V \xrightarrow{\pi} U, D_\pi)$ is acyclic

if: $\begin{cases} \bullet\ M \setminus V : \text{compact} \\ \bullet\ \ker(D_{\pi,b}) = 0 \quad \forall b \in U. \end{cases}$

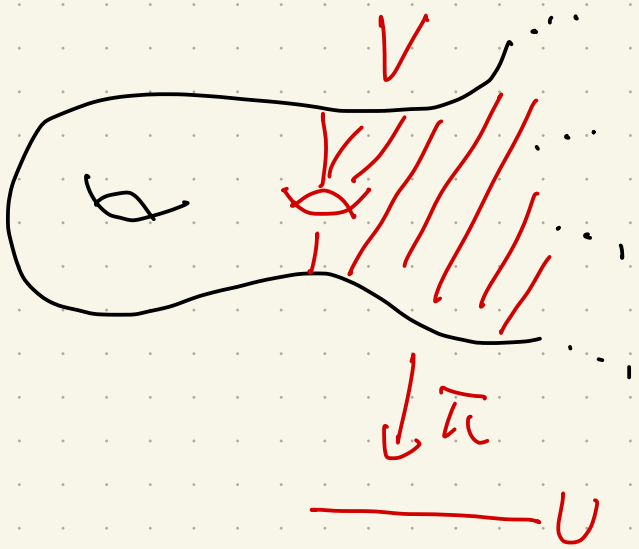
op. on $\pi^{-1}(b)$ (closed!!)



§4. FFY's formulation

Def. $(V \xrightarrow{\pi} U, D_\pi)$ is **acyclic** if:
- $M \setminus V$: compact
- $\ker(D_{\pi,b}) = 0 \quad \forall b \in U$.

op. on $\pi^{-1}(b)$ (closed!!)

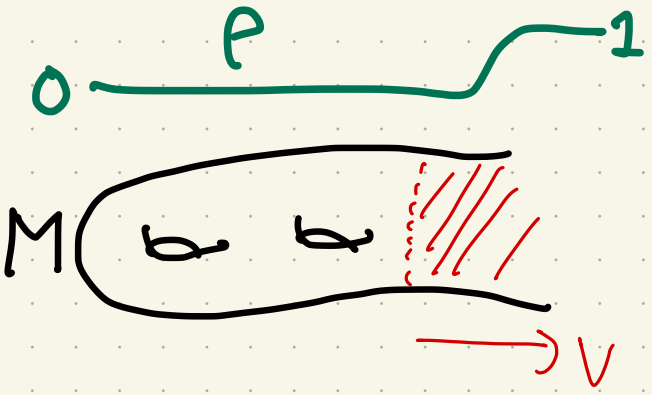


Thm. If $(V \xrightarrow{\pi} U, D_\pi)$: acyclic and $V = M$, then

$$\ker(D + tD_\pi)^2 \left(= \ker(D + tD_\pi)\right) = 0 \quad \forall t \gg 0.$$

KEY
- $DD_\pi + D_\pi D$: diff. op. along fibers.
- $\ker(D_{\pi,b}) = 0 \Rightarrow |DD_\pi + D_\pi D| \leq {}^\exists C_b D_\pi^2$ (apriori estimate)

§4. FFY's formulation



General case : $D_t := D + t\rho D_\pi$

ρ : cut-off fct.

Thm. If $(V \xrightarrow{\pi} U, D_\pi)$: acyclic, then :

$$\mathrm{ind}(D_t) = \dim \ker_{L^2}(D_t^+) - \dim \ker_{L^2}(D_t^-) < \infty$$

$$\| \quad !!$$

$$\mathrm{ind}(M, V \xrightarrow{\pi} U, D_\pi) = \mathrm{ind}(M, V) : \text{excision property.}$$



$$\mathrm{ind}(M, V) = \mathrm{ind}(M', V')$$

§4. FFY's formulation

Rem.

- π : diffeo $\Longrightarrow$ $D_\pi (= h)$ recovers Witten's deform.
- $\dim \mathrm{Ker}(D_t) < \infty$ itself is non-trivial.
- $(V \xrightarrow{\pi} U, D_\pi)$ provides a formulation of index theory on non-cpt. mfd.
- Non-complete case.
- M : closed $\Longrightarrow \mathrm{ind}(D_t) = \overbrace{\mathrm{ind}(D)}^{\text{topological inv}} =: \mathrm{ind}(M:W)$

§3. FFY's formulation

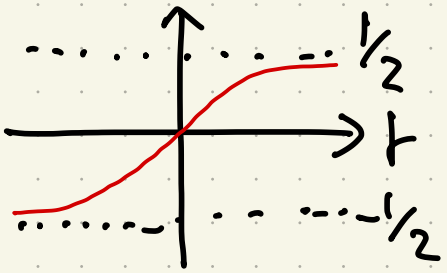
Ex. $\underbrace{W = \mathbb{C}^2 \times M \longrightarrow M}_{\text{Clifford mod. b'dle}} = \overbrace{S^1 \times \mathbb{R} \longrightarrow \mathbb{R}}^{S^1\text{-b'dle}}$

(θ, r)

$$D := \begin{pmatrix} 0 & \underbrace{-\partial_\theta - \sqrt{-1}\,\partial_r + \sqrt{-1}\,\sigma}_{D^-} \\ \underbrace{\partial_\theta - \sqrt{-1}\,\partial_r - \sqrt{-1}\,\sigma}_{D^+} & 0 \end{pmatrix} : \Gamma(W) \circlearrowleft$$

$$D_\pi := \begin{pmatrix} 0 & -\partial_\theta + \sqrt{-1}\,\sigma \\ \partial_\theta - \sqrt{-1}\,\sigma & 0 \end{pmatrix}$$

σ :

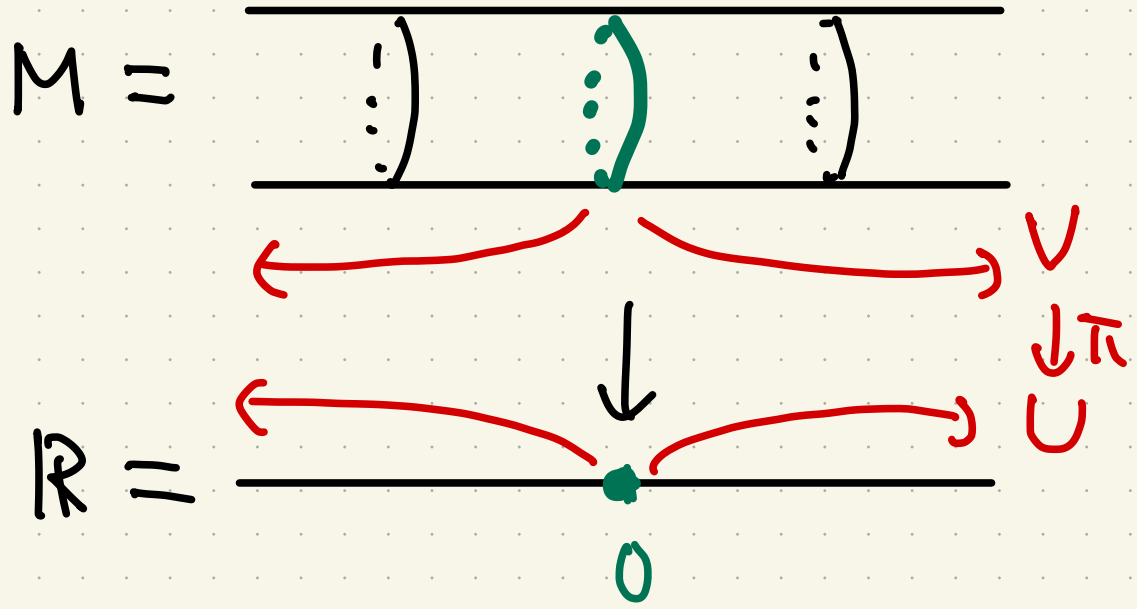


$$\rightsquigarrow \operatorname{Ker}(D_{\pi,r}) = \begin{cases} 0 & (r \neq 0) \\ \mathbb{C} & (r = 0) \end{cases}$$

$\sigma(r) \in \mathbb{Z} \Longleftrightarrow r = 0$

$V := S^1 \times (\mathbb{R} \setminus \{0\}) \rightsquigarrow$ acyclic !!

§3. FFY's formulation



$\rightsquigarrow \operatorname{Ker}_{L^2}(D_t^+) \cong \mathbb{C}, \quad \operatorname{Ker}_{L^2}(D_t^-) = 0$

$\rightsquigarrow \operatorname{ind}(M, V) = 1.$

↳ captures "lattice points" in $M \setminus V$.

§5. Symplectic case

§5. Symplectic case

Basic setup

- $(L, \nabla) \xrightarrow{\iota} (M, \omega) \xrightarrow{\pi} B \supset BS$

  closed symp. mfd.

  pre-quantize line bdle ($curv(\nabla) = -\sqrt{-1}\omega$)

  Lagrangian fibration (w/o. sing. fib)

  $= \{b \in B \mid \Gamma^{par}(L|_{\pi^{-1}(b)}) \neq 0\}$

  fiber = flat torus (Arnold-Liouville)

- Fix ω-compati $J : TM \to TM$.

  (not necessarily Kähler)

  - $W = \Lambda^\bullet T^*_{\mathbb{C}} M^{0,1} \otimes L \cong \Lambda^\bullet \overline{T^*M} \otimes L$
  - $D = D_L = \bar{\partial}_L + \bar{\partial}_L^*$: Dolbeault-Dirac.

§5. Symplectic case

Key observation

$$W|_{\pi^{-1}(b)} = \Lambda^\bullet \overline{T^*M} \otimes L|_{\pi^{-1}(b)}$$

$$\cong \Lambda^\bullet \left(T^*\pi^{-1}(b) \oplus J T^*\pi^{-1}(b) \right) \otimes L|_{\pi^{-1}(b)}$$

$$= \Lambda^\bullet T^*\pi^{-1}(b) \otimes L|_{\pi^{-1}(b)} = \underset{\text{of } \pi^{-1}(b)}{\text{de Rham}} \otimes L|_{\pi^{-1}(b)}$$

flat b'dle

$\circlearrowleft$

D_π, $D_\pi|_{\pi^{-1}(b)} = d_{L,b} + d_{L,b}^* =: D_{\pi,b}$

(deRham op $\otimes L|_{\pi^{-1}(b)}$)

$$\rightsquigarrow \ker(D_{\pi,b}) \cong H^*(\pi^{-1}(b), L|_{\pi^{-1}(b)}) = 0$$

$$\iff H^0(\pi^{-1}(b), L|_{\pi^{-1}(b)}) = \Gamma^{\text{par}}(L|_{\pi^{-1}(b)}) = 0$$

$$\iff b \notin BS$$

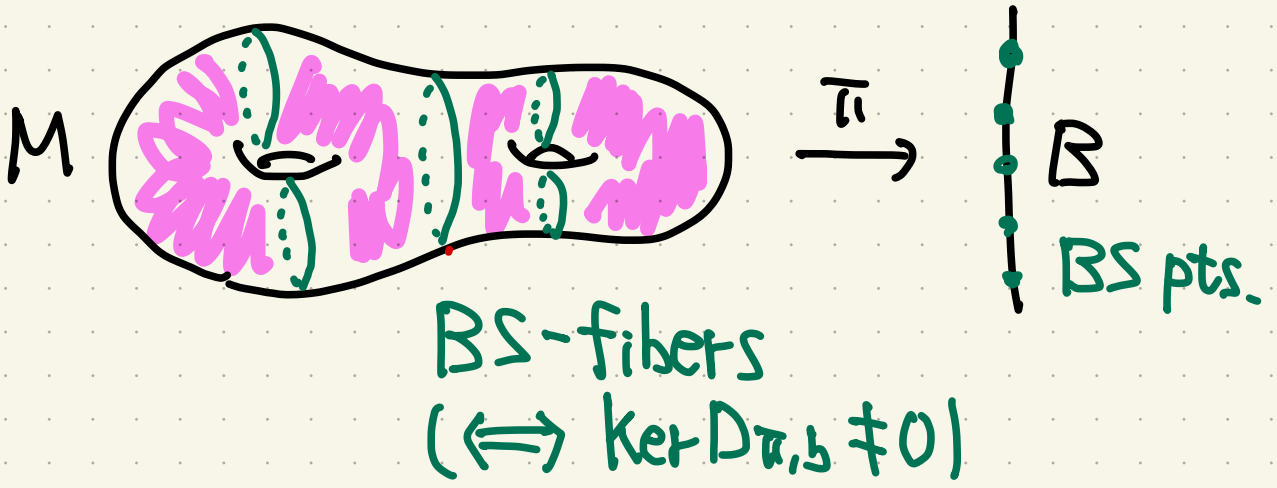
§5. Symplectic case

Key observation $\Rightarrow (V \xrightarrow{\pi} U, D_\pi)$: acyclic gen. real pol.

$\pi^{-1}(U)$ $\quad$ $B \setminus$ BS pts.

$$\Rightarrow \mathrm{ind}(D + tD_\pi) = \mathrm{ind}(D)$$

localizes to BS-fibers !!



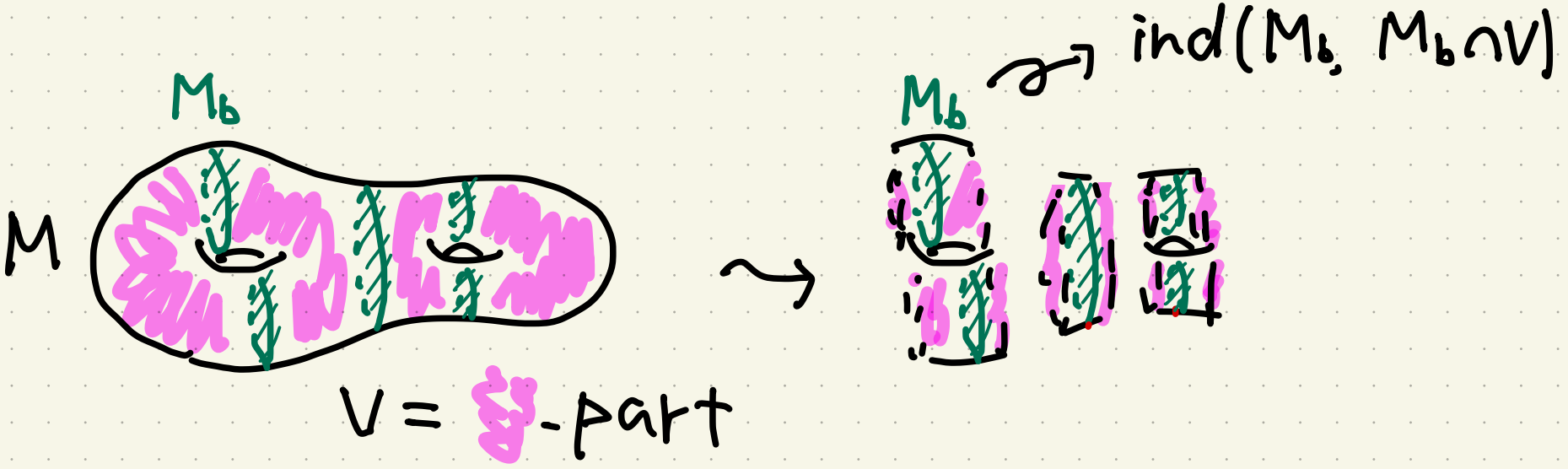
No contribution from ▨-part to ind(D) !!

§5. Symplectic case

Thm. (FFY)

$$\mathrm{ind}(D) = \sum_{b \in BS} [\text{local contribution from nbd. } M_b \text{ of } \pi^{-1}(b)]$$

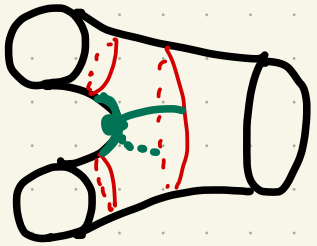
$$= \sum_{b \in BS} \mathrm{ind}(M_b, M_b \cap V)$$



§6. Comments

§6. Comments

$\mathrm{ind}(M_b, M_b \cap V) = ??$

- $= 1$ for non-singular $\pi^{-1}(b)$ $(\cong T^n)$.
- $= 1$ for toric-type (elliptic) singular fiber $(\cong T^{k \le n})$.
- $= 0$ or -1 for 
- $= 1$ for $S^n \subset TS^n (\supset TS^n \setminus S^n)$ $(n \ge 1)$.
  w.r (periodic) Ham. action of $\|\cdot\|$.

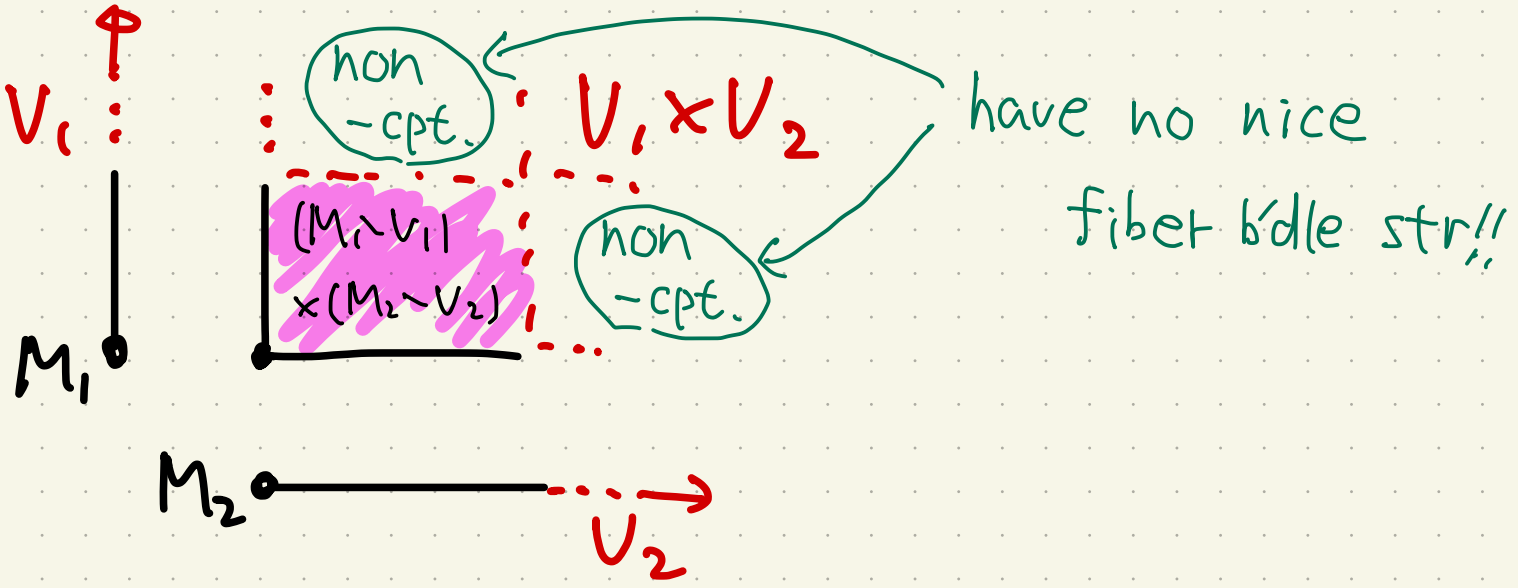
§6. Comments

Product formula ?

- M_i with $V_i \xrightarrow{\pi_i} U_i$
  $\cap$
  M_i, $M_i \setminus V_i$: cpt

$(i=1,2)$

Take ⇝ product



Need a refinement...

§6. Comments

Def. (compatible fibration)

$\{\pi_\alpha : V_\alpha \longrightarrow U_\alpha\}_{\alpha\in A}$ is a compatible fib. on V

$:\overset{\text{def}}{\Longleftrightarrow}$

i) $V = \bigcup_{\alpha\in A} V_\alpha$

ii) $\forall \alpha \in A \left[\pi_\alpha : V_\alpha \longrightarrow U_\alpha : \underbrace{\text{fibet b'dle}}_{\text{(closed mfd.)}}\right.$

iii) $\forall \alpha,\beta \in A \left[V_\alpha \cap V_\beta = \pi_\alpha^{-1}(\pi_\alpha(V_\alpha \cap V_\beta)) = \pi_\beta^{-1}(\pi_\beta(V_\alpha \cap V_\beta))\right]$

ex. toric mfd., Hamiltonian torus mfd.

§6. Comments

- FFY. I : prototype

$(\pi : V \longrightarrow U, D_\pi)$

- FFY. II : index theory for

$\{(\pi_\alpha : V_\alpha \longrightarrow U_\alpha), D$

- FFY. III : application to $[Q, R] = 0$.

J. Math. Sci. Univ. Tokyo
17 (2010), 1–26.

Torus Fibrations and Localization of Index I – Polarization and Acyclic Fibrations

By Hajime Fujita, Mikio Furuta* and Takahiko Yoshida†

Abstract. We define a local Riemann-Roch number for an open symplectic manifold when a completely integrable system without Bohr-Sommerfeld fiber is provided on its end. In particular when a structure of a singular Lagrangian fibration is given on a closed symplectic manifold, its Riemann-Roch number is described as the sum of the number of nonsingular Bohr-Sommerfeld fibers and a contribution of the singular fibers. A key step of the proof is formally explained as a version of Witten's deformation applied to a Hilbert bundle.

1. Introduction

The purpose of this paper is to give a localization technique for the index of spin^c Dirac operator. Our localization makes use of no group action, but a family of acyclic flat connections on tori. A typical example is given by a symplectic manifold with an integrable system.

For a completely integrable system on a closed symplectic manifold, the Riemann-Roch number is sometimes equal to the number of Bohr-Sommerfeld fibers. For Kähler case D. Borthwick, T. Paul and A. Uribe gave a relationship between Bohr-Sommerfeld Lagrangians and the kernels of twisted spin^c Dirac operators using Fourier integral operators [2]. A problem here would be how to count singular Bohr-Sommerfeld fibers, and how to count the contribution from other singular fibers if there are. M. D. Hamilton and E. Miranda dealt with the singular Bohr-Sommerfeld fibers using J. Śniatycki's framework and found a discrepancy between quantizations via real and complex polarizations under this framework [10, 11].

*Partially supported by JSPS Grant 19340015, 19204003 and 16340015.
†Partially supported by Advanced Graduate Program in Mathematical Sciences at Meiji University, JSPS Grant-in-Aid for Young Scientists (B) 20740029, and Fujyukai Foundation.
2000 *Mathematics Subject Classification.* Primary 53D50; Secondary 58J20.
Key words: Geometric quantization, index theory, localization.

§6. Comments

- FFY. I : prototype

$$(\pi : V \longrightarrow U, D_\pi)$$

- FFY. II : index theory for

$$\{(\pi_\alpha : V_\alpha \longrightarrow U_\alpha), D_\alpha\}_{\alpha \in A}$$

- FFY. III : application to $[Q, R] = 0$.

Commun. Math. Phys. 326, 585–633 (2014)
Digital Object Identifier (DOI) 10.1007/s00220-014-1890-7

Communications in Mathematical Physics

Torus Fibrations and Localization of Index II

Local Index for Acyclic Compatible System

Hajime Fujita[1], **Mikio Furuta**[2], **Takahiko Yoshida**[3]

[1] Department of Mathematical and Physical Sciences, Japan Women's University, 2-8-1 Mejirodai, Bunkyo-ku, Tokyo 112-8681, Japan. E-mail: fujitah@fc.jwu.ac.jp
[2] Graduate School of Mathematical Sciences, The University of Tokyo, 3-8-1 Komaba, Meguro-ku, Tokyo 153-8914, Japan. E-mail: furuta@ms.u-tokyo.ac.jp
[3] Department of Mathematics, School of Science and Technology, Meiji University, 1-1-1 Higashimita, Tama-ku, Kawasaki 214-8571, Japan. E-mail: takahiko@meiji.ac.jp

Received: 30 September 2010 / Accepted: 27 November 2013
Published online: 20 February 2014 – © Springer-Verlag Berlin Heidelberg 2014

Abstract: We give a framework of localization for the index of a Dirac-type operator on an open manifold. Suppose the open manifold has a compact subset whose complement is covered by a family of finitely many open subsets, each of which has a structure of the total space of a torus bundle. Under an acyclic condition we define the index of the Dirac-type operator by using the Witten-type deformation, and show that the index has several properties, such as excision property and a product formula. In particular, we show that the index is localized on the compact set.

Contents

1. Introduction . . . 585
2. Summary . . . 587
3. An Index Theory for Complete Riemannian Manifolds . . . 588
4. Perturbation and Local Index . . . 592
5. The Case of Hamiltonian Torus Actions . . . 597
6. Compatible Fibration and Acyclic Compatible System . . . 605
7. Local Index and Localization Formula . . . 610
8. Product Formula of Local Indices . . . 612
9. Four-Dimensional Case . . . 616
A. Supplements to Sect. 3 . . . 627
B. Proof of Lemma 5.3 . . . 632
References . . . 633

1. Introduction

This paper is the second of the series concerning a localization of the index of elliptic operators.

§6. Comments

- FFY. I : prototype

  $(\pi : V \longrightarrow U, D_\pi)$

- FFY. II : index theory for

  $\{(\pi_\alpha : V_\alpha \longrightarrow U_\alpha),$ [

- FFY. III : application to $[Q, R] = 0$.

Commun. Math. Phys. 326, 585–633 (2014)
Digital Object Identifier (DOI) 10.1007/s00220-014-1890-7

Communications in Mathematical Physics

Torus Fibrations and Localization of Index II

Local Index for Acyclic Compatible System

Hajime Fujita[1], **Mikio Furuta**[2], **Takahiko Yoshida**[3]

[1] Department of Mathematical and Physical Sciences, Japan Women's University, 2-8-1 Mejirodai, Bunkyo-ku, Tokyo 112-8681, Japan. E-mail: fujitah@fc.jwu.ac.jp
[2] Graduate School of Mathematical Sciences, The University of Tokyo, 3-8-1 Komaba, Meguro-ku, Tokyo 153-8914, Japan. E-mail: furuta@ms.u-tokyo.ac.jp
[3] Department of Mathematics, School of Science and Technology, Meiji University, 1-1-1 Higashimita, Tama-ku, Kawasaki 214-8571, Japan. E-mail: takahiko@meiji.ac.jp

Received: 30 September 2010 / Accepted: 27 November 2013
Published online: 20 February 2014 – © Springer-Verlag Berlin Heidelberg 2014

Abstract: We give a framework of localization for the index of a Dirac-type operator on an open manifold. Suppose the open manifold has a compact subset whose complement is covered by a family of finitely many open subsets, each of which has a structure of the total space of a torus bundle. Under an acyclic condition we define the index of the Dirac-type operator by using the Witten-type deformation, and show that the index has several properties, such as excision property and a product formula. In particular, we show that the index is localized on the compact set.

Contents

1. Introduction . . . 585
2. Summary . . . 587
3. An Index Theory for Complete Riemannian Manifolds . . . 588
4. Perturbation and Local Index . . . 592
5. The Case of Hamiltonian Torus Actions . . . 597
6. Compatible Fibration and Acyclic Compatible System . . . 605
7. Local Index and Localization Formula . . . 610
8. Product Formula of Local Indices . . . 612
9. Four-Dimensional Case . . . 616
A. Supplements to Sect. 3 . . . 627
B. Proof of Lemma 5.3 . . . 632
References . . . 633

1. Introduction

This paper is the second of the series concerning a localization of the index of elliptic operators.



ご清聴
ありがとう
ございました。

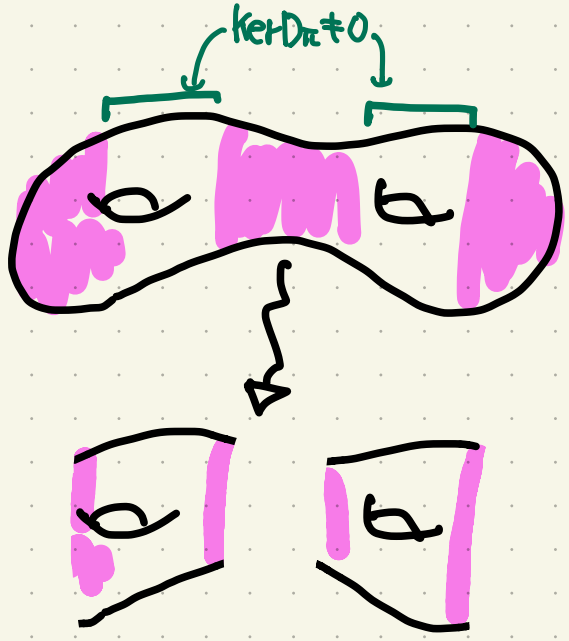




$$\mathrm{ind}(D) = \mathrm{ind}(D + tD_\pi)$$

↓

Localization
at "$\ker D_\pi \neq 0$"





古田先生は、
とてもお茶目で
よく笑う人です。





$$\mathrm{ind}(D) = \mathrm{ind}(D + tD_\pi)$$

↓

Localization at "$\ker D_\pi \neq 0$"

